THEORETICAL DESCRIPTION AND EXPERIMENTAL EVALUATION OF THE EFFECT OF THE INTERROGATION OSCILLATOR FREQUENCY NOISE ON THE STABILITY OF A PULSED ATOMIC FREQUENCY STANDARD.

G. Santarelli, Ph. Laurent and A. Clairon.
BNM/Laboratoire Primaire du Temps et des Fréquences,
Observatoire de Paris, 61, Av. de l'Observatoire, 75014 Paris, France.
Giorgio.Santarelli@obspm.fr

G. J. Dick, C. A. Greenhall.
Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr., Pasadena, CA 911109, U.S.A.

C. Audoin.
Laboratoire de l'Horloge Atomique, Unité Propre du CNRS associée à l'Université de Paris Sud, Bat. 221, Université de Paris Sud, 91405 Orsay Cédex, France.

ABSTRACT.

In this paper we evaluate the effect of the interrogation oscillator frequency noise on the stability of a pulsed atomic frequency standard such as an atomic fountain or ion trap frequency standard. The atomic response to a phase perturbation in Ramsey and multi-Rabi interrogation schemes has been calculated using the density matrix formalism. The results of these calculations are used to obtain a simple model for the limitation of the frequency standard stability. An experimental evaluation of this effect has been performed by using the Cs atomic fountain frequency standard. Possible means to reduce this effect are considered.

1 INTRODUCTION.

The development of new passive frequency standards using trapped ions or cold atoms has produced devices with a potential fractional frequency stability of the order of $10^{-13}\tau^{-1/2}$ or better. In this new type of standards, the internal interrogation process and the control of the interrogation oscillator are periodic, with period T_c. The frequency of this oscillator is compared to that of the atomic resonance during a part of the operating cycle only and its frequency is controlled at the end of each cycle.
In the late eighties, one of us [1] at the Jet Propulsion Laboratory derived the atomic response to the oscillator frequency fluctuations using a geometrical approach. Furthermore, he has shown that the oscillator frequency noise at Fourier frequencies which are close to multiples of $1/T_c$ is down-converted, leading to a degradation of the long-term frequency stability.
This work has been the motive for new thoughts on this subject and the present paper reports briefly a part of the results achieved.
The atomic response to a frequency variation in the interrogation oscillator is deduced from a quantum mechanical treatment for different interrogation schemes using the Ramsey and the multi-Rabi [2] methods.
The equation giving the frequency stability limitation arising from the oscillator frequency noise down-conversion is derived from simple reasoning. The validity of this formula is verified experimentally. Finally, possible means of reducing this spurious effect are considered.

2 SENSITIVITY FUNCTION TO OSCILLATOR FREQUENCY FLUCTUATIONS: A RECALL.

Let δP be the change of the probability that a transition occurred, at the outcome of the atom interaction with the R.F. field. If this change is the result of a fluctuation, $\delta\omega(t)$, of the frequency of the field during the interaction, we have [1]:

$$\delta P = \frac{1}{2} \int_{\text{int.}} g(t)\, \delta\omega(t)\, dt \tag{1}$$

This equation defines $g(t)$, the sensitivity function to frequency fluctuations. The integration holds during the time T_i of the interaction.
It is worth noting that the R.F. field is frequency modulated. Its (angular) frequency is $\omega(t)+\omega_m$ or $\omega(t)-\omega_m$ according to the half period of modulation considered. $\omega(t)$ is very close to the atom resonance frequency ω_0 and ω_m is the modulation depth. In Eq. (1), $\delta\omega(t)$ is the fluctuation of $\omega(t)$ and $g(t)$ is the sensitivity function at the frequency $\omega_0\pm\omega_m$.
It has been shown that $g(t)$ can be calculated by introducing an infinitesimally small phase step ε at time t in the oscillator frequency. It produces the change $\delta P(t,\varepsilon)$ of the probability that a transition occurred and $g(t)$ is given by:

$$g(t) = 2 \lim_{\varepsilon\to 0} \delta\, P(t,\varepsilon) / \varepsilon \tag{2}$$

In frequency standards considered here the function $g(t)$ it not a constant during each cycle (T_c). As explained in the following this causes the degradation of the frequency stability of the locked oscillator.

European Frequency Time Forum, 5-7 March 1996 Conference Publication No 418 © IEE 1996

3 FUNDAMENTAL LIMITATION OF THE FREQUENCY STABILITY DUE TO SAMPLING.

3.1 Origin of the effect.

The control loop being closed, frequency corrections are applied to the interrogation oscillator at discrete times t_k, at the end of each cycle. Immediately after this adjustment, the frequency offset is $\Delta\omega_s(t_k)$. Between the instants t_k and t_{k+1}, the frequency of the oscillator varies freely and the offset $\Delta\omega(t)=\omega(t)-\omega_0$ of the R.F. field is given by :

$$\Delta\omega(t) = \Delta\omega_s(t_k) + \Delta\omega_f(t) - \Delta\omega_f(t_k) \quad t_k < t < t_{k+1} \quad (3)$$

The subscripts s and f stand for slaved and free, respectively. The frequency offsets considered are those of the R.F. field when the deviation due to the modulation is not taken into consideration.

The variation $\Delta\omega(t)$ of the frequency of the interrogation field induces a change in the probability that the transition occurred and this adds a term to the error signal. This term can be interpreted as being produced by a false frequency offset $\delta\omega_f(t_k)$ of the applied field, which is given by :

$$\delta\omega_f(t_k) = \frac{1}{g_0 T_c}\int_{t_k}^{t_k+T_c} g(t)\left[\Delta\omega_f(t) - \Delta\omega_f(t_k)\right]dt \ ,(4)$$

where g_0 is the mean value of $g(t)$ during the same cycle. The frequency control loop includes a numerical integration. It can be shown that its equation is the following :

$$\begin{aligned}&\Delta\omega_s(t_k) - (1-\beta)\Delta\omega_s(t_{k-1}) + \beta\,\Delta\omega_s(t_{k-2})\\ &= \Delta\omega_f(t_k) - \Delta\omega_f(t_{k-1}) - \beta\left[\delta\omega_f(t_{k-1}) + \delta\omega_f(t_{k-2})\right]\end{aligned} \quad (5)$$

where the optical detection noise is not considered. and β is the open loop gain. For slow frequency fluctuations, Eq. (5) shows that we have :

$$\Delta\omega_s(t_k) = -\delta\omega_f(t_k) \quad (6)$$

In Eq.(4), $g(t)/g_0$ is a periodic function of time with frequency $1/T_c$. Therefore the spectral components of $\Delta\omega_f(t)$ around frequencies m/T_c will be translated to the very low frequencies. This is the equivalent of an aliasing phenomenon in a sampling process.

3.2 Derivation of the formula for the long term frequency stability limitation.

We will consider the limitation of the frequency stability, for observation times τ sufficiently larger than the servo-loop time constant. For this analysis we assume that the bandwidth of the down-converted noise, Δf, is of the order of magnitude of $1/\tau$, and thus smaller than $1/T_c$.

The low frequency noise in the bandwidth Δf originates from spectral components of the oscillator noise around Fourier frequencies m/T_c. Then, it suffices to consider that part of the oscillator frequency noise which is filtered in a set of spectral windows centred around frequencies m/T_c and having a noise bandwidth $2\Delta f$. The Rice representation [3] of this narrow band limited noise is the following :

$$\Delta\omega_f^F(t) = \sum_{m=1}^{\infty}\left[p_m(t)\sin 2\pi m\frac{t-t_k}{T_c} + q_m(t)\cos 2\pi m\frac{t-t_k}{T_c}\right] \quad (7)$$

where $p_m(t)$ and $q_m(t)$ are slowly variable random amplitudes. The constant phase, $-2\pi m t_k/T_c$ is introduced for convenience and does not affect the final result. The one-sided power spectral density of $p_m(t)$ and $q_m(t)$ is related to that of $\Delta\omega_f(t)$ around m/T_c by:

$$S_{p_m}(f \le \Delta f) = S_{q_m}(f \le \Delta f) = 2S_{\Delta\omega_f}(m/T_c) \quad (8)$$

The frequency offset $\delta\omega_f(t)$ is calculated by substituting $\Delta\omega_f^F(t)$ for $\Delta\omega_f(t)$ in Eq. (4). According to the assumptions made, $p_m(t)$ and $q_m(t)$ vary very little during the time interval T_c. Therefore, they can be taken out of the integral sign and we obtain :

$$\delta\omega_f(t_k) = -\Delta\omega_f^F(t_k) + \frac{1}{g_0}\sum_{m=1}^{\infty}\left[g_m^s p_m(t_k) + g_m^c q_m(t_k)\right] \quad (9)$$

with

$$\begin{pmatrix} g_m^s \\ g_m^c \end{pmatrix} = \int_0^{T_c} g(\theta)\begin{pmatrix}\sin 2\pi m\theta / T_c \\ \cos 2\pi m\theta / T_c\end{pmatrix} d\theta \quad (10)$$

where $g(\theta)$ is defined during the time interval $[0,T_c]$.

Referring to Eq. (6), the second term of the right-hand side of Eq. (9) represents the low frequency noise of the slaved oscillator resulting from the aliasing. The power spectral density of these fluctuations is easily derived from Eqs. (8) and (9), and the related Allan variance is given by:

$$\sigma_{y\,\mathrm{lim}}^2(\tau) = \frac{1}{\tau}\sum_{m=1}^{\infty}\left(\frac{g_m^{c\,2}}{g_0^2} + \frac{g_m^{s\,2}}{g_0^2}\right) S_y^f(m/T_c) \quad (11)$$

where $S_y^f(m/T_c) = S_{\Delta\omega f}^f(m/T_c)/\omega_0^2$ is the one-sided power spectral density of the relative frequency fluctuations of the free running interrogation oscillator at Fourier frequencies m/T_c.

4 EXAMPLE OF THE FREQUENCY SENSITIVITY FUNCTION AND OF THE LEVEL OF FREQUENCY STABILITY LIMITATION.

4.1 Principle of the calculation of *g(t)*.

As shown in [4], the change of the quantum state of the atoms interacting with a R.F. field of given amplitude and phase can be represented in a matrix form. We have, in general :

$$\begin{pmatrix} a_1(\theta) \\ a_2(\theta) \\ a_3(\theta) \end{pmatrix} = R \begin{pmatrix} a_1(0) \\ a_2(0) \\ a_3(0) \end{pmatrix} \qquad (12)$$

where a_1 and a_2 denote the atomic coherence and a_3 the population difference of the two levels involved in the transition. The column matrices at the right and at the left represent the atom properties at the beginning and at the end of an interaction of duration θ, respectively. R is a 3x3 matrix whose elements depend on θ, the R.F. amplitude and the phase.

Therefore, the population difference of atoms submitted to various amplitude and phase conditions during their interaction with the magnetic R.F. field can be calculated from matrix products. The effect of the small phase step ε occurring at a given instant during the interaction can be expressed easily. The subsequent change $\delta P(t,\varepsilon)$ of the probability that a transition took place during the interaction follows and $g(t)$ is obtained using Eq. (2). In the event that the R.F. amplitude is not a constant during the interaction, two different methods can be implemented. One may divide the interaction time into elementary intervals during which the amplitude is assumed a constant. Or, the differential equations describing the evolution of a_1, a_2 and a_3 (see Eq. 5.2.20 of [4]) may be integrated numerically.

4.2 Ramsey method of interrogation

This method is applied in the cesium fountain at LPTF [5]. The atoms experience successively two R.F. fields, each for the time interval τ, and there is no R.F. field applied between these two partial interactions, for the time T. Assuming $T \gg \tau$ and that the magnetic R.F. field is a constant, represented by the Rabi frequency b, we have :

$$g(\theta) = \begin{cases} 0 & 0 \le \theta \le t_p \\ d \sin b\theta & t_p \le \theta \le tp + \tau \\ d \sin b\tau & t_p + \tau \le \theta \le t_p + T + \tau \\ d \sin b(T + 2\tau - \theta) & t_p + T + \tau \le \theta \le t_p + T + 2\tau \\ 0 & t_p + T + 2\tau \le \theta \le T_c \end{cases} \qquad (13)$$

where d=-$\sin\Omega_0 T \sin b\tau$ and $\Omega_0 = \pm\, \omega_m$ according to the half period of modulation considered. Fig. 1 shows the variation of $g(\theta)$ (assumed centred during the cycle, solid line) and of the related coefficients g_m/g_0, for $b\tau=\pi/2$ which provides the optimal condition of interrogation. It is assumed τ=0.015s, T=0.5s and T_c=1s. The function $g(\theta)$ for $b\tau=3\pi/2$ is shown also (dashed line).

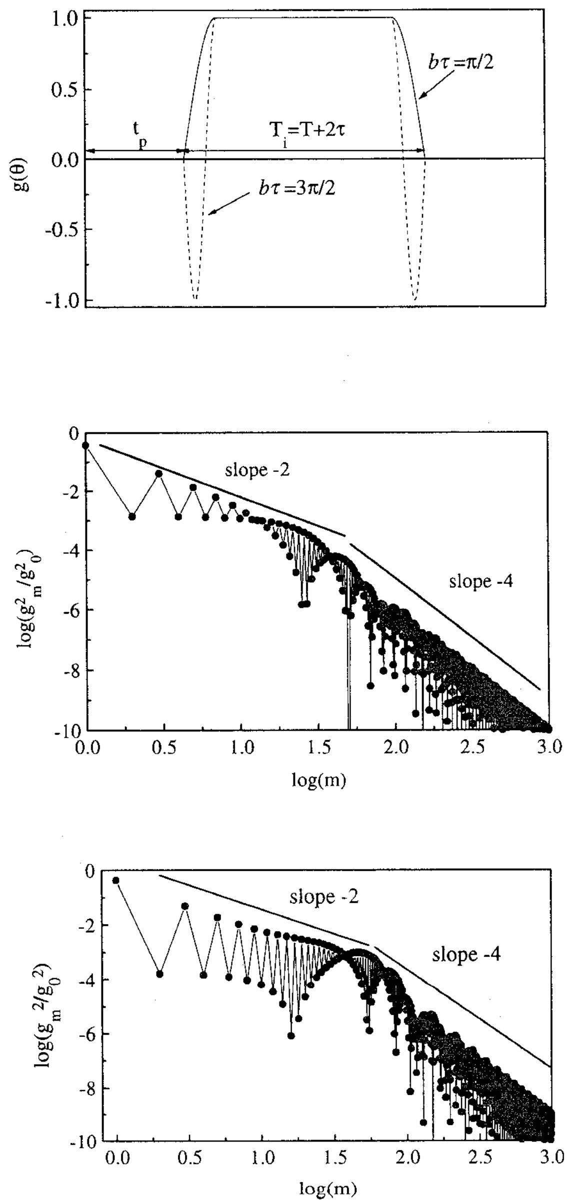


Fig. 1 The function $g(\theta)$ and the related spectrum in the case of a Ramsey interrogation scheme for $b\tau=\pi/2$ and $b\tau=3\pi/2$.

In order to assess the level of the frequency stability limitation due to the aliasing, we assume that the oscillator is a VCXO and that the power spectral density of its relative frequency fluctuations is given by:

$S_y^f(f)$ =3.2 10^{-29} f^2+1.0 10^{-27} f+3.2 10^{-26}/f (14)

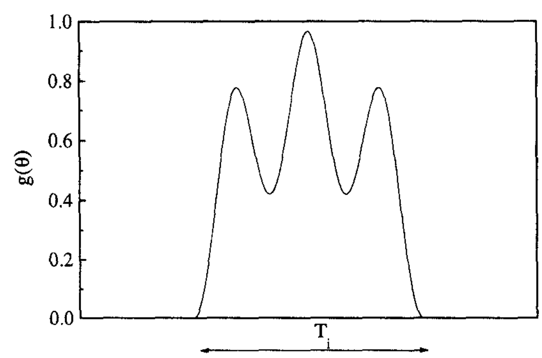
This hypothetical oscillator shows a flicker floor at the level 2.1 10^{-13}.
For the Ramsey method of interrogation, we have:

$\sigma_{y,\lim}(\tau)$=1.17 $10^{-13}\tau^{-1/2}$ (15)

for the given values of the parameters.

4.3 Interrogation in a TE_{01n} cavity

In the PHARAO project [6], it is planned to launch balls of cold cesium atoms along the axis of a cylindrical cavity tuned to a TE_{01n} resonant mode. During their interrogation, the atoms experience a magnetic R.F. field whose amplitude is proportional to $\sin(n\pi\theta/T_i)$, where T_i is the total interaction time. In that case, $g(t)$ is calculated numerically.



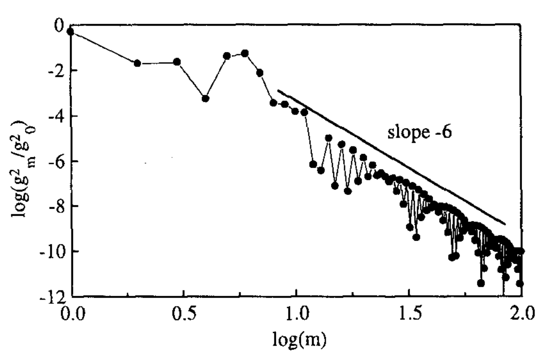


Fig. 2 The function $g(\theta)$ and the related spectrum in the case of a multi-Rabi interrogation scheme.

Fig. 2 shows, for n=3, the variation of $g(\theta)$ (assumed centred during the cycle) and of the related coefficients g_m/g_0. Here, the operating parameters are chosen so as to provide the steepest slope of the resonance curve. This is achieved for b_cT_i/n = 3.66 and ω_mT_i = 2.31, where b_c is the Rabi frequency at an anti-node of the magnetic field.
It is worth noting that, for m larger than about 10, $(g_m/g_0)^2$ decreases as m^{-6}. This property provides a very good immunity against the white phase noise of the oscillator. For T_i = 0,53 s and T_c=1s, we have:

$\sigma_{y\lim}(\tau)$=1.35 10^{-13} $\tau^{-1/2}$ (15)

which is similar to the value given in Eq. (13).

5. EXPERIMENTAL EVALUATION OF THE FREQUENCY STABILITY DEGRADATION.

In order to verify the model and provide evidence of the aliasing effect, we have made various measurements with an oscillator voluntarily degraded with different types of frequency noise. For this ,we use the LPTF Cs atomic fountain standard [5].

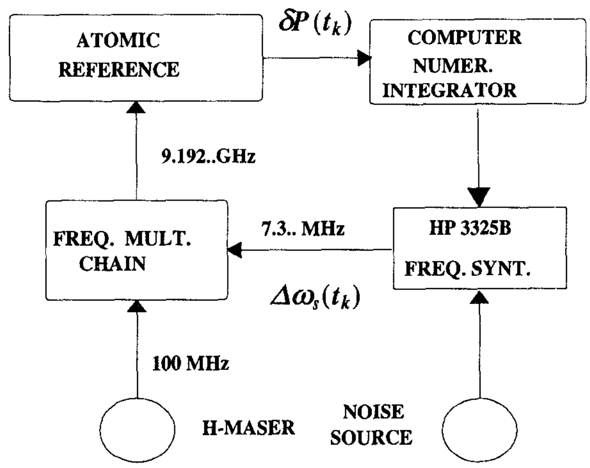


Fig. 3 Schematic of the atomic fountain frequency

We have used three different sources of noise. Firstly a white noise voltage in the range 0.1 Hz-1600 Hz (f^0) with the possibility of using different low-pass filters. Secondly a flicker noise generator f^{-1}in the range 0.5-100 Hz and thirdly a generator with spectral density proportional to f^{-3}, for Fourier frequencies from 0.5 to 100 Hz .

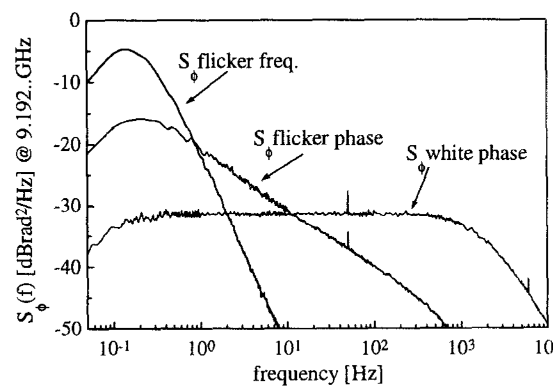


Fig. 4 Measured phase noise spectral densities of the degraded interrogation oscillator.

We use the noise generators to drive the phase modulation input of a HP3325B synthesiser. The

HP 3325B is used as an external generator for the fountain frequency multiplication chain (Fig. 3). The phase noise added to the HP 3325B is transferred to the interrogation oscillator spectrum. The added noise is high-pass filtered (0.1Hz) to avoid the degradation of the medium and long-term stability of the oscillator (Fig. 4).

The Allan standard deviation of the locked oscillator is calculated taking the frequency output of the synthesiser at the end of each cycle. As shown in Fig. 5 the fractional frequency stability of the free running oscillator, behaves as τ^{-1} whereas the stability of the locked oscillator is proportional to $\tau^{-1/2}$. This clearly shows that the frequency stability of the locked oscillator is dominated by the aliasing noise for integration times longer than 10-20s.

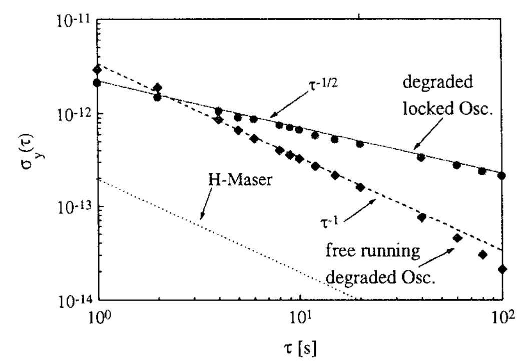


Fig. 5 Fractional frequency stability of the free running and locked oscillator

We measured the stability for two conditions: $b\tau=\pi/2$ and $b\tau=3\pi/2$. Tables 1 and 2 report the calculated (Eq. (11)) and measured values for the flicker phase and flicker frequency noises.

Table 1, $b\tau=\pi/2$

Type of noise	$\sigma_y meas.(1s)$	$\sigma_y calc.(1s)$
Frequency Flicker	$2.4\ 10^{-12}$	$2.3\ 10^{-12}$
Phase Flicker	$3.0\ 10^{-12}$	$2.9\ 10^{-12}$

Table 2, $b\tau=3\pi/2$

Type of noise	$\sigma_y meas.(1s)$	$\sigma_y calc.(1s)$
Frequency Flicker	$2.8\ 10^{-12}$	$2.4\ 10^{-12}$
Phase Flicker	$4.8\ 10^{-12}$	$4.6\ 10^{-12}$

For these types of noise the frequency stability limitation mainly depends on the ratio between interrogation time and cycle time. In order to verify precisely the model we need a measurement which is more sensitive to the shape of $g(\theta)$. In the case of white phase noise the frequency noise spectrum behaves as f^{+2} and consequently the down-conversion is strongly dependent on the harmonic content of $g(\theta)$. The possibility to change the frequency noise spectrum by low-pass filtering the noise source with different cut-off frequencies improves the sensitivity of the measurements. Fig. 6 reports the calculated and measured results, which agree to the limit of the measurements errors (20%). Measurements performed with the filtered white phase noise confirms the validity of the model even for different interrogation oscillator levels ($b\tau=\pi/2$ and $3\pi/2$)

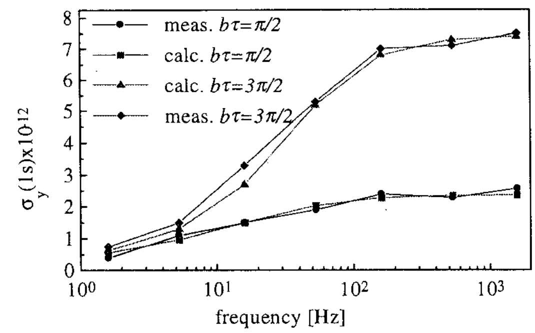


Fig. 6 Measured and calculated frequency stability versus the cut-off frequency of the white phase noise.

5. POSSIBLE MEANS FOR THE MINIMIZATION OF THE FREQUENCY STABILITY LIMIT

In a cesium fountain, in the PHARAO set-up [6] or in a trapped ions frequency standard, it will not be possible to use notch filters to reject the oscillator noise at Fourier frequencies m/T_c since T_c is equal to 1s or less. The level of the annoying effect would obviously be reduced with an oscillator, such as a cryogenic sapphire oscillator [7] showing a much improved spectral purity. However, we will limit ourselves here to exemplify the beneficial effect of the increase of the interrogation duty cycle [1] or in the case of cold atoms frequency standards of the release of several balls during each cycle, assuming that the VCXO characterised by Eq. (14) is slaved to the atomic resonator. In the case of the Ramsey method of interrogation, the total interrogation time is $T_i=T+2\tau$. Fig. 7 shows the variation of σ_{ylim} versus T_i/T_c for three sets of parameter values.

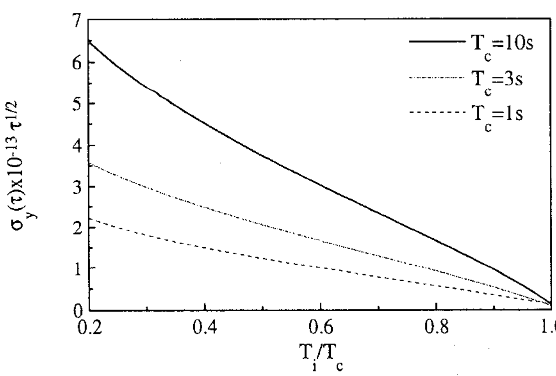


Fig. 7 Frequency stability versus the interrogation duty cycle for the Ramsey interrogation.

Several balls of cold cesium atoms can be launched during a cycle. Fig. 8 shows, as an example, the variation of σ_{ylim} versus the number of balls. It is assumed that the interrogation occurs in a TE_{013} cavity and that the time interval between two successive ball releases is equal to $T_i/6$. Again, three sets of parameter values are considered.

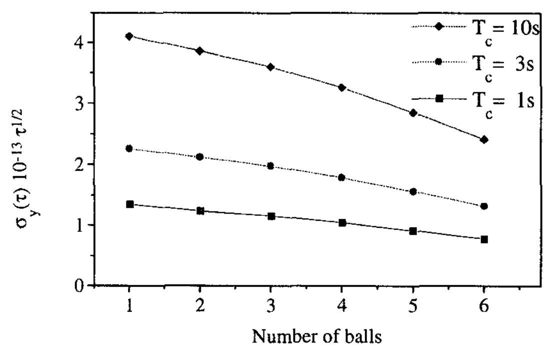


Fig. 8 Calculated frequency stability versus the number of launched atomic balls for the multi-Rabi case.

Thus, it is possible, in an atomic resonator based on the interrogation of atoms launched sequentially, to minimise the limiting effect considered. This may be accomplished by a proper design of the resonator leading to a duty cycle as large as possible and/or by launching a cloud of atoms several times during a cycle. In the case of the ion traps it has been proposed to use two parallel traps [8].

One may also note that in the two given examples, σ_{ylim} is the smallest for $1/T_c$=1Hz. This value is the closest to the Fourier frequency, f_Q, for which $S_y^f(f)$ of the VCXO, shows a minimum. With the data of Eq. (14), we have f_Q=5Hz. This suggests that, whenever possible, the characteristics of the atomic resonator and of the VCXO should be matched. This is achieved when the condition $f_Q \approx 1/T_c$ is fulfilled.

CONCLUSIONS.

In this paper we have developed a simple model for the degradation of the frequency stability in a pulsed atomic frequency standard due to aliasing of the frequency noise of the interrogating oscillator. We have also compared results of the calculations based on this model with experimental values obtained by using the LPTF Cs atomic fountain with a voluntary degraded oscillator. The theory and the experiments agree in the limits of the measurement errors. The frequency stability limitation is about 10^{-13} if we use a state of the art 5-10 MHz VCXO BVA. Better results can be achieved using cryogenic sapphire oscillators. The model shows that the limitation comes primarily from the flicker frequency noise of the oscillator and the typical white phase floor does not affect the results. It seems that there are no obvious signal processing techniques which would be able to reduce the consequences of the detrimental aliasing.

ACKNOWLEDGMENTS.

This work forms a part of the program of scientific developments of the Bureau National de Metrologie (BNM). The research programme has the financial and logistic support of the French Space Agency (CNES) via the Pharao project. We also acknowledge the financial participation of the Conseil Général de la Région Ile de France through the Sesame programme.

REFERENCES.

1. Dick G. J., 1987, "Local oscillator induced instabilities in trapped ion frequency standards," in Proc. of *Precise Time and Time Interval*, Redendo Beach, 133-147.

2. De Marchi A., Bava E., Tavella P., 1993, "Accuracy analysis of the atom-microwave interaction in a TE_{017} cylindrical Rabi Cavity for a high C-field cesium beam resonator," in Proc. of *7th European Frequency and Time Forum*, Neuchatel, 489-493.

3. Rice S.O., 1945, "Mathematical analysis of random noise". Bell System Tech. J., 23, 1944, 282-332 and 24, , 46-156.

4. Vanier J. and Audoin C., 1989, "The quantum physics of atomic frequency standards", Adam Hilger Ed.

5. Clairon A., Laurent Ph., Santarelli G., Ghezali S., Lea S.N., and Bahoura M., 1995, "A cesium fountain frequency standard: preliminary results," IEEE Trans. Instrum. Meas., 44, 128-131.

6. Clairon A., Laurent Ph., Nadir A., Drewsen M., Grison D., Lounis B. and Salomon C.. 1992, "A simple and compact source of cold atoms for cesium fountains and microgravity clocks" in Proc. of 6th European Frequency and Time Forum, Noordwijk (The Netherlands), 27-33.

7. Luiten A. N., A. G. Mann, M. E. Costa and Blair D. G., 1995, "Power stabilized cryogenic sapphire oscillator," IEEE Trans. Instrum. Meas., 44, 132-135.

8. Dick G. J., Prestage J. D., Greenhall C. A., and Maleki L., 1990, "Local oscillator induced degradation of medium-term stability in passive atomic frequency standards," in Proc. 22nd Ann. Precise Time and Time Interval (PTTI), 487-508.